

The Nature of Space Time

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Abstract

We first examine the approximation involved in the conventional differentiable spacetime manifold. We then analyse how, going beyond this approximation, we reach the non commutative spacetime of recent approaches. It is shown that this provides the rationale for El Naschie's transfinite Cantorian spacetime. The nature and form of the consequent Generalized Uncertainty Principle is also briefly investigated.

1 Introduction

All of Classical Physics and Quantum Theory, is based on the Minkowski spacetime, as for example in the case of Quantum Field Theory, or Reimannian spacetime as in the case of General Relativity. In the non relativistic theories, Newtonian spacetime, is used, which is a special case of Minkowskian spacetime. But in all these cases the common denominator is that we are dealing with a differentiable manifold.

This breaks down however in Quantum Gravity, String Theory and more recent approaches, be it at the Planck scale, or at the Compton scale[1, 2, 3, 4]. The underlying reason for this breakdown of a differentiable spacetime manifold is the Uncertainty Principle— as we go down to arbitrarily small spacetime intervals, we encounter arbitrarily large energy momenta. As Wheeler put it[5], "no prediction of spacetime, therefore no meaning for spacetime is

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the verdict of the Quantum Principle. That object which is central to all of Classical General Relativity, the four dimensional spacetime geometry, simply does not exist, except in a classical approximation.” Before proceeding to analyse the nature of spacetime beyond the classical approximation, let us first analyse briefly the nature of classical spacetime itself.

2 The ”Classical” Approximation

We can get an insight into the nature of the usual spacetime by considering the well known formulation of Quantum Theory in terms of stochastic processes [6, 7, 8, 9]. This will also facilitate subsequent considerations.

In the stochastic or Nelsonian theory, we deal with a double Wiener process which leads to a complex velocity $V - iU$. It is this complex velocity that leads to Quantum Theory from the usual diffusion theory (Cf.[7] for details). To see this in a simple way, let us write the usual diffusion equation as

$$\Delta x \cdot \Delta x = \frac{h}{m} \Delta t \equiv \nu \Delta t \quad (1)$$

Equation (1) can be rewritten as the usual Quantum Mechanical relation,

$$m \frac{\Delta x}{\Delta t} \cdot \Delta x = h = \Delta p \cdot \Delta x \quad (2)$$

We are dealing here, with phenomena within the Compton or de Broglie wavelength.

We now treat the diffusion constant ν to be very small, but non vanishing. That is, we consider the semi classical case. This is because, a purely classical description, does not provide any insight.

It is well known that in this situation we can use the WKB approximation. In this case the right hand side of the representation of the Nelsonian wave function,

$$\psi = \sqrt{\rho} e^{i/\hbar S}$$

goes over to, in the one dimensional case, for simplicity,

$$(p_x)^{-\frac{1}{2}} e^{\frac{1}{\hbar} \int p(x) dx}$$

so that we have, on comparison,

$$\rho = \frac{1}{p_x} \quad (3)$$

ρ being the probability density. In this case the condition $U \approx 0$, that is, the velocity potential becoming real, implies

$$\nu \cdot \nabla \ln(\sqrt{\rho}) \approx 0 \quad (4)$$

This semi classical analysis suggests that $\sqrt{\rho}$ is a slowly varying function of x , infact each of the factors on the left side of (4) would be $\sim 0(\hbar)$, so that the left side is $\sim 0(\hbar^2)$ (which is being neglected). Then from (3) we conclude that p_x is independent of x , or is a slowly varying function of x . The equation of continuity now gives

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = \frac{\partial \rho}{\partial t} = 0$$

That is the probability density ρ is independent or nearly so, not only of x , but also of t . We are thus in a stationary and homogenous scenario. This is strictly speaking, possible only in a single particle universe, or for a completely isolated particle, without any effect of the environment. Under these circumstances we have the various conservation laws and the time reversible theory, all this taken over into Quantum Mechanics as well. This is an approximation valid for small, incremental changes, as indeed is implicit in the concept of a differentiable space time manifold.

Infact the well known displacement operators of Quantum Theory which define the energy momentum operators are legitimate and further the energy and momenta are on the same footing only under this approximation[10].

We would now like to point out the well known close similarity between the Nelsonian formulation mentioned above (Cf.(1) and (2) and the hydrodynamical formulation for Quantum Mechanics, which also leads to identical equations on writing the wave function as above. These two approaches were reconciled by considering quantized vortices at the Compton scale (Cf.[11, 9]). To proceed further, we start with the Schrodinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \quad (5)$$

Remembering that for momentum eigen states we have, for simplicity, for one dimension

$$\frac{\hbar}{i} \frac{\partial}{\partial x} \psi = p \psi \quad (6)$$

where p is the momentum or p/m is the velocity v , we take the derivative with respect to x (or \vec{x}) of both sides of (5) to obtain, on using (6),

$$i\hbar \frac{\partial(v\psi)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2(v\psi) + \frac{\partial V}{\partial x} \psi + V v \psi \quad (7)$$

We would like to compare (7) with the well known equation for the velocity in hydrodynamics[12], following from the Navier-Stokes equation,

$$\rho \frac{\partial v}{\partial t} = -\nabla p - \rho \alpha T g + \mu \nabla^2 v \quad (8)$$

In (8) v is a small perturbational velocity in otherwise stationary flow between parallel plates separated by a distance d , p is a small pressure, ρ is the density of the fluid T is the temperature proportional to $Q(z)v$, μ is the Navier-stokes coefficient and α is the coefficient of volume expansion with temperature. Also required would be

$$\beta \equiv \frac{\Delta T}{d}.$$

v itself is given by

$$v_z = W(z) \exp(\sigma t + i k_x x + i k_y y), \quad (9)$$

z being the coordinate perpendicular to the fluid flow.

We can now see the parallel between equations (7) and (8). To verify the identification we would require that the dimensionless Rayleigh number

$$R = \frac{\alpha \beta g d^4}{\kappa \nu}$$

should have an analogue in (7) which is dimensionless, κ, ν being the thermometric conductivity and viscosity.

Remembering that

$$\nu \sim \frac{\hbar}{m}$$

and

$$d \sim \lambda$$

where λ is the Compton wavelength in the above theory (Cf.[9] for details) and further we have

$$\rho \propto f(z)g = V \quad (10)$$

for the identification between the hydrostatic energy and the energy V of Quantum Mechanics, it is easy using (10) and earlier relations to show that the analogue of R is

$$(c^2/\lambda^2) \cdot \lambda^4 \cdot (m/h)^2 \quad (11)$$

The expression (11) indeed is dimensionless and of order 1. Thus the mathematical identification is complete.

Before proceeding, let us look at the physical significance of the above considerations (Cf.[13] for a graphic description.) Under conditions of stationery flow, when the difference in the temperature between the two plates is negligible there is total translational symmetry, as in the case of the displacement operators of Quantum Theory. But when there is a small perturbation in the velocity (or equivalently the temperature difference), then beyond a critical value the stationarity and homogeneity of the fluid is disrupted, or the symmetry is broken and we have the phenomena of the formation of Benard cells, which are convective vortices and can be counted. This infact is the "birth" of space (Cf.[13] for a detailed description).

In the context of the above identification, the Benard cells would correspond to the formation of quantized vortices, which latter had been discussed in detail in the literature (Cf.[9] and [14]). This transition would correspond to the "formation" of spacetime. As discussed in detail in [9] these quantized vortices can be identified with elementary particles, in particular the electrons. Interestingly, Einstein himself considered electrons as condensates from a background electromagnetic field[15].

However in order to demonstrate that the above quantized vortex formation is not a mere mathematical analogy, we have to show that the critical value of the wave number k in the expression for the velocity in the hydrodynamical flow (9) is the same as the critical value of the quantized vortex length. In terms of the dimensionless wave number $k' = k/d$, this critical value is given by[12]

$$k'_c \sim 1$$

In the case of the quantized vortices at the Compton scale l , remembering that d is identified with l itself we have,

$$l'_c(\equiv)k'_c \sim 1,$$

exactly as required.

In this connection it may be mentioned that due to fluctuations in the Zero Point Field or the Quantum vacuum, there would be fluctuations in the metric given by[5]

$$\Delta g \sim l_P/l$$

where l_P is the Planck length $\sim 10^{-33}cms$ and l is a small interval under consideration. At the same time the fluctuation in the curvature of space would be given by

$$\Delta R \sim l_P/l^3$$

Normally these fluctuations are extremely small but as discussed in detail elsewhere[16], this would imply that at the Compton scale of a typical elementary particle $l \sim 10^{-11}cms$, the fluctuation in the curvature would be ~ 1 . This is symptomatic of the formation of what we have termed above as quantized vortices.

Further if a typical time interval between the formation of such quantized vortices which are the analogues of the Benard cells is τ , in this case the Compton time, then as in the theory of the Brownian Random Walk[17], the mean time extent would be given by

$$T \sim \sqrt{N}\tau \tag{12}$$

where N is the number of such quantized vortices or elementary particles (Cf.also [9, 11]). It is quite remarkable that the equation (12) holds good for the universe itself because T the age of the universe $\sim 10^{17}secs$ and N the number of elementary particles $\sim 10^{80}$, τ being the Compton time $\sim 10^{-23}secs$. Interestingly, this nature of time would automatically make it irreversible, unlike the conventional model in which time is reversible.

It may be mentioned that an equation similar to (12) can be deduced by the same arguments for space extension also and this time we get the well known Eddington formula viz.,

$$R \sim \sqrt{N}l \tag{13}$$

where R is the radius of the universe and l is the Compton wavelength. Further starting from (12) one can work out a whole scheme of what may be called fluctuational cosmology, in which not just the Eddington formula (13) above, but also all the other supposedly mysterious and inexplicable large number relations of Dirac and the Weinberg formula relating the mass of the pion to the Hubble Constant can be deduced theoretically. Furthermore, this cosmology predicts an ever expanding and accelerating universe, as is now recognised to be the case (Cf.[18, 9] for details).

Once we recognize the minimum space time extensions, then we immediately are lead to an underlying non commutative geometry given by[9],

$$[x, y] = 0(l^2), [x, p_x] = i\hbar[1 + 0(l^2)], [t, E] = i\hbar[1 + 0(\tau^2)] \quad (14)$$

As was shown a long time ago, relations like (14) are Lorentz invariant. At this stage we recognise the nature of spacetime as given by (14) in contrast to the stationary and homogeneous spacetime discussed earlier. Witten[19] has called this Fermionic spacetime as contrasted to the usual spacetime, which he terms Bosonic. Indeed one could show the origins of the Dirac equation of the electron from (14). We could also argue that (14) provides the long sought after reconciliation between electromagnetism and gravitation[20, 21]. The usual differentiable spacetime geometry can be obtained from (14) if l^2 is neglected, and this is the approximation that has been implicit.

3 Cantorian Spacetime and Metric

Thus spacetime is a collection of such cells or elementary particles very much in the spirit of El Naschie's Cantorian spacetime[22, 23, 24]. As pointed out earlier, this spacetime emerges from a homogeneous stationary non spacetime when the symmetry is broken, through random processes. The question that comes up then is, what is the metric which we use? This has been touched upon earlier, and we will examine it again.

We first makes a few preliminary remarks. When we talk of a metric or the distance between two "points" or "particles", a concept that is implicit is that of topological "nearness" - we require an underpinning of a suitably large number of "open" sets[25]. Let us now abandon the absolute or background space time and consider, for simplicity, a universe (or set) that consists solely of two particles. The question of the distance between these

particles (quite apart from the question of the observer) becomes meaningless. Indeed, this is so for a universe consisting of a finite number of particles. For, we could isolate any two of them, and the distance between them would have no meaning. We can intuitively appreciate that we would in fact need distances of intermediate or more generally, other points.

In earlier work[26], motivated by physical considerations we had considered a series of nested sets or neighbourhoods which were countable and also whose union was a complete Hausdorff space. The Urysohn Theorem was then invoked and it was shown that the space of the subsets was metrizable. The argument went something like this.

In the light of the above remarks, the concepts of open sets, connectedness and the like reenter in which case such an isolation of two points would not be possible.

More formally let us define a neighbourhood of a particle (or point or element) A of a set of particles as a subset which contains A and at least one other distinct element. Now, given two particles (or points) or sets of points A and B , let us consider a neighbourhood containing both of them, $n(A, B)$ say. We require a non empty set containing at least one of A and B and at least one other particle C , such that $n(A, B) \subset n(A, C)$, and so on. Strictly, this "nested" sequence should not terminate. For, if it does, then we end up with a set $n(A, P)$ consisting of two isolated "particles" or points, and the "distance" $d(A, P)$ is meaningless.

We now assume the following property[26]: Given two distinct elements (or even subsets) A and B , there is a neighbourhood N_{A_1} such that A belongs to N_{A_1} , B does not belong to N_{A_1} and also given any N_{A_1} , there exists a neighbourhood $N_{A_{\frac{1}{2}}}$ such that $A \subset N_{A_{\frac{1}{2}}} \subset N_{A_1}$, that is there exists an infinite topological closeness.

From here, as in the derivation of Urysohn's lemma[25], we could define a mapping f such that $f(A) = 0$ and $f(B) = 1$ and which takes on all intermediate values. We could now define a metric, $d(A, B) = |f(A) - f(B)|$. We could easily verify that this satisfies the properties of a metric.

With the same motivation we will now deduce a similar result, but with different conditions. In the sequel, by a subset we will mean a proper subset, which is also non null, unless specifically mentioned to be so. We will also consider Borel sets, that is the set itself (and its subsets) has a countable covering with subsets. We then follow a pattern similar to that of a Cantor

ternary set [25, 27]. So starting with the set N we consider a subset N_1 which is one of the members of the covering of N and iterate this process so that N_{12} denotes a subset belonging to the covering of N_1 and so on.

We note that each element of N would be contained in one of the series of subsets of a sub cover. For, if we consider the case where the element p belongs to some $N_{12\dots m}$ but not to $N_{1,2,3\dots m+1}$, this would be impossible because the latter form a cover of the former. In any case as in the derivation of the Cantor set, we can put the above countable series of sub sets of sub covers in a one to one correspondence with suitable sub intervals of a real interval (a, b) .

Case I

If $N_{1,2,3\dots m} \rightarrow$ an element of the set N as $m \rightarrow \infty$, that is if the set is closed, we would be establishing a one to one relationship with points on the interval (a, b) and hence could use the metric of this latter interval, as seen earlier.

Case II

It is interesting to consider the case where in the above iterative countable process, the limit does not tend to an element of the set N , that is set N is not closed and has what we may call singular points. We could still truncate the process at $N_{1,2,3\dots m}$ for some $m > L$ arbitrary and establish a one to one relationship between such truncated subsets and arbitrarily small intervals in a, b . We could still speak of a metric or distance between two such arbitrarily small intervals.

This case is of interest because of recent work which describes elementary particles as, what may be called Quantum Mechanical Kerr-Newman Black Holes or vortices, where we have a length of the order of the Compton wavelength as seen in the previous section, within which spacetime as we know it breakdown. Such cut offs as seen lead to a non commutative geometry (14) and what may be called fuzzy spaces[28, 29, 4].(We note that the centre of the vortex is a singular point). In any case, the number of particles in the universe is of the order 10^{80} , which approximates infinity from a physicist's point of view.

Interestingly, we usually consider two types of infinite sets - those with cardinal number n corresponding to countable infinities, and those with cardinal number c corresponding to a continuum, there being nothing inbetween. This is the well known but unproven Continuum Hypotheses.

What we have shown with the above process is that it is possible to conceive an intermediate possibility with a cardinal number $n^p, p > 1$.

We also note again the similarity with El Naschie's transfinite Cantor sets. In the above considerations three properties are important: the set must be closed i.e. it must contain all its limit points, perfect i.e. in addition each of its points must be a limit point and disconnected i.e. it contains no nonnull open intervals. Only the first was invoked in Case I.

4 The Generalized Uncertainty Principle

In theories of Quantum Gravity and also String Theory we encounter what may be called the Generalized Uncertainty Principle

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha \cdot \frac{\Delta p}{\hbar} \quad (15)$$

This is symptomatic of the non zero spacetime extension, and indeed also follows from (14). This could be construed to imply a correction to the velocity of light as has been noted in the literature. It could also be taken to be a correction to the Einstein mass-energy formula $\sim 0(l^2)$ (Cf.ref.[9] for details).

A more complete picture emerges from the following simple model of a one dimensional lattice, the points being spaced a length l apart. In this case the energy, as is known, can be shown to be given by

$$E \sim mc^2 \cos(\alpha l) \quad (16)$$

where α is proportional to the wave number.

A comparison with results following from (14) or (15) shows that the latter are truncated versions of (16), truncated $\sim 0(l^2)$.

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